Another perspective on S^1_{fil}

Goal: understand $\mathrm{fil}^* HH(A/k)$, $\mathrm{fil}^* HC^-(A/k)$

$$gr^i \simeq \wedge^i L_{A/k}[i] \qquad gr^i \simeq \widehat{dR}^{\geq i}_{A/k}[2i]$$

Analogy

$A^{\otimes_k S^1} \simeq HH(A/k)$	$\mathrm{Map}_{(\mathrm{CAlg}_k^\Delta)^{BS^1}}(HH(A/k), B) \simeq \mathrm{Map}_{\mathrm{CAlg}_k^\Delta}(A, B)$
$\Omega^\bullet_{A/k}$	$\mathrm{Map}_{\mathrm{cdga}_k^{str}}(\Omega^\bullet_{A/k}, B^\bullet) \simeq \mathrm{Map}_{\mathrm{CAlg}_k^0}(A, B^0)$

Issues (1) ∞-categorical vs 1-categorical

(2) no analogue of grading in HH univ prop.

Strategy (1') Formulate universal "dg simpl comm alg"

$$L\Omega^\bullet_{A/k} = \left\{ A \xrightarrow{d} L_{A/k} \to \Lambda^2 L_{A/k} \to \quad \right\}$$

(2') Formulate compatibility of HKR filtration on HH w/ simpl comm alg struct + S^1-action, characterize $\mathrm{fil}^* HH$ as

"filtered simpl comm alg w/ filtered S^1-action"

M-R-T version: (2') $[L_{fil} X / S^1_{fil}] \longrightarrow BS^1_{fil}$

relative affine derived scheme

(1') associated graded of thm

Today: another perspective

Warm-up

let G finite abelian group. regard it as $\mathrm{Spec}(k^G)$ — bicommutative Hopf algebra

$$\mathrm{QCoh}(BG) \simeq \mathrm{Mod}_{k[G]} \simeq \mathrm{Comod}_{k^G} \qquad (M \xrightarrow{\text{module map}} M \otimes k^G)$$

$$\mathrm{Aff}_{BG} \simeq \qquad \simeq \underbrace{\mathrm{Comod}_{k^G}}_{\text{coalg in } \mathrm{CAlg}_k^\Delta}(\mathrm{CAlg}_k^\Delta) \qquad (A \xrightarrow{\text{alg map}} A \otimes k^G)$$

Less of a warm up

$G \curvearrowright S^1$

$$\mathrm{QCoh}(BS^1) \simeq \mathrm{Mod}_{k[S^1]} \simeq \mathrm{Comod}_{k^{S^1}}$$

$C_*(S^1;k) \qquad C^*(S^1;k)$

want to say: $\mathrm{Aff}_{BS^1} \simeq \mathrm{Comod}_{k^{S^1}}(\mathrm{CAlg}_k^{\Delta})$

issue: k^{S^1} not **connective**, not in CAlg_k^{Δ}

<u>Solution</u> (Mathew, Bhatt)

"nonconnective simpl comm rings" = <u>derived rings</u>

$$\begin{tikzcd}
\mathrm{CAlg}_k^{\Delta} \arrow[r, hook] \arrow[d, "\text{monadic (LSym monad)}"'] & \mathrm{DAlg}_k \arrow[r] \arrow[d] & E_\infty\text{-alg}_k \arrow[dl] \\
\mathrm{Mod}_k^{cn} \arrow[r, hook] & \mathrm{Mod}_k &
\end{tikzcd}$$

<u>idea</u> extend LSym to Mod_k.

<u>e.g.</u> limit of a diagram in CAlg_k^{Δ}. better answer in DAlg_k k^X $R\Gamma(X; \mathcal{O}_X)$

<u>Upshot</u> can consider comodules over k^{S^1} in DAlg_k

$$\wr$$

coalg in DAlg_k

Not a warm-up

$S^1 \rightsquigarrow S^1_{fil}$, derived ring $k^{S^1} \rightsquigarrow$ filtered derived ring $k^{S^1_{fil}}$

$(p : S^1_{fil} \to \mathbb{A}^1/\mathbb{G}_m . \ p_* \mathcal{O})$

Shall have:

$$QCoh(BS^1_{fil}) \simeq \mathrm{Comod}_{k^{S^1_{fil}}}$$

$$\mathrm{Aff}_{BS^1_{fil}} \simeq \mathrm{Comod}_{k^{S^1_{fil}}}(\mathrm{FilDAlg}_k)^{cr}$$

What is $k^{S^1_{fil}}$?

underlying k^{S^1}

assoc gr $k[\eta]$

$\rightsquigarrow$ $k^{S^1_{fil}}$ must be Postnikov filtration (at filtered E_∞-algebra)

Thm Let $A \in \mathrm{DAlg}_k$ coconnective, then $\tau_{\geq *} A$ is canonically a filtered derived ring; moreover this determines an embedding $\tau_{\geq *} : \mathrm{DAlg}_k^{\mathrm{ccn}} \longrightarrow \mathrm{FilDAlg}_k$.

Cor $\tau_{\geq *}(k^{S^1})$ canonically a filtered derived bialgebra (coalgebra in $\mathrm{FilDAlg}_k$)

$\rightsquigarrow$ $k^{S^1}_{\mathrm{fil}} = \tau_{\geq *}(k^{S^1})$, $\mathrm{Fil}_{S^1}\mathrm{DAlg}_k := \mathrm{Comod}_{k^{S^1}_{\mathrm{fil}}}(\mathrm{FilDAlg}_k)$

(<u>Need</u> $\mathrm{LSym}^i(k[-n])$ is $(-in)$-connective)

<u>Thm</u> $\mathrm{fil}^*_{HKR}\, HH(A/k)$ underlies $HH_{fil}(A/k) \in \mathrm{Fil}_{\leq 1}\mathrm{DAlg}_k$ characterized by

$$\mathrm{Map}_{\mathrm{Fil}_{\leq 1}\mathrm{DAlg}_k}(HH_{fil}(A/k), B) \simeq \mathrm{Map}_{\mathrm{DAlg}_k}(A, \mathrm{und}(B))$$

How to think about filtered S^1 action?

$$k^{S^1_{fil}} = \tau_{\geq *}(k^{S^1}) \quad \text{is dual to} \quad k[S^1_{fil}] \simeq \tau_{\geq *}(k[S^1])$$

$$\mathrm{Comod}_{k^{S^1_{fil}}} \simeq \mathrm{Mod}_{k[S^1_{fil}]}$$

$$\begin{gathered} \wr \\ k[S^1] \leftarrow k[1] \leftarrow 0 \leftarrow 0 \leftarrow \end{gathered}$$

"S^1 action increasing filtration degree"

E.g. $X \in \mathrm{Mod}_k^{BS^1} \simeq \mathrm{Mod}_{k[S^1]}$

$$\rightsquigarrow \quad \tau_{\geq *}(X) \simeq \mathrm{Mod}_{\tau_{\geq *} k[S^1]}$$

<u>Rmk</u> on assoc graded, recover "graded Λ-modules"

I like $\mathbb{D}_+$ instead of Λ, $(k[\varepsilon])$

graded $\mathbb{D}_+$-modules $\simeq$ "homotopy coherent cochain complexes" $X^\bullet$ with diff of the form

$$X^i[1] \longrightarrow X^{i+1}$$

$\mathbb{D}_+^\vee = gr(k^{S^1_{fil}})$ is a graded derived bialgebra

$\rightsquigarrow \quad DG_+DAlg_k := \mathrm{Comod}_{\mathbb{D}_+^\vee}(GrDAlg_k)$

Thm $\{\wedge^i L_{A/k}[-i]\}$ underlie $L\Omega^{+\bullet}_{A/k}$ in $DG_+ DAlg_k$

characterized by $\mathrm{Map}_{DG_+ DAlg_k}(L\Omega^{+\bullet}_{A/k}, B^\bullet)$

$$\simeq \mathrm{Map}_{DAlg_k}(A, B^0).$$

Universal property $\xRightarrow{\text{formally}}$ $gr(HH_{fil}(A/k)) \simeq L\Omega^{+\bullet}_{A/k}$

$$\mathrm{Sym}^i\left(L^{\mathrm{top}}_{A_k}[1]\right)$$